Exercise 7.8.2

A particular solution to $y' = y^2/x^3 - y/x + 2x$ is $y = x^2$. Find a more general solution.

Solution

Since one solution is known, the Riccati equation can be transformed into a Bernoulli equation by making the substitution $y = x^2 + u$. Differentiating both sides of the substitution with respect to x gives $y' = 2x + u'$.

$$y' = \frac{y^2}{x^3} - \frac{y}{x} + 2x \quad \rightarrow \quad \cancel{2x} + u' = \frac{(x^2+u)^2}{x^3} - \frac{(x^2+u)}{x} + \cancel{2x} \quad \rightarrow \quad u' = \frac{u^2}{x^3} + \frac{u}{x}$$

Bring u/x to the left side.

$$u' - \frac{u}{x} = \frac{u^2}{x^3}$$

Divide both sides by u^2.

$$u^{-2}u' - \frac{1}{x}u^{-1} = \frac{1}{x^3}$$

Make the substitution $p = u^{-1}$. Then $p' = -u^{-2}u'$ by the chain rule.

$$-p' - \frac{1}{x}p = \frac{1}{x^3}$$

Multiply both sides by -1.

$$p' + \frac{1}{x}p = -\frac{1}{x^3}$$

This is a first-order linear ODE that can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^x \frac{1}{s}\,ds\right) = e^{\ln x} = x$$

Proceed with the multiplication.

$$xp' + p = -\frac{1}{x^2}$$

The left side can be written as $d/dx(Ip)$ by the product rule.

$$\frac{d}{dx}(xp) = -\frac{1}{x^2}$$

Integrate both sides with respect to x.

$$xp = \frac{1}{x} + C$$

Divide both sides by x.

$$p(x) = \frac{1}{x^2} + \frac{C}{x}$$

Now that the ODE is solved, change back to u.

$$u^{-1} = \frac{1}{x^2} + \frac{C}{x}$$

Invert both sides.

$$\begin{aligned} u(x) &= \frac{1}{\frac{1}{x^2} + \frac{C}{x}} \cdot \frac{x^2}{x^2} \\ &= \frac{x^2}{1 + Cx} \end{aligned}$$

Therefore, since $y = x^2 + u$,

$$y(x) = x^2 + \frac{x^2}{1 + Cx}.$$